Hello everyone,

This week we will be continuing our study of logic from week 1 by learning about proofs -- a method to show that a statement is true or false. It may help to quickly review Chapter 1. There are 5 types of proofs we will be learning about and each of them have their own characteristics and properties that make them a "better fit" to solve particular types of problems. I have categorized them below (somewhat subjectively). I will show a few examples of when to use a particular type of proof method:

1. Direct proof -- We use a direct proof: when a very simple example or counterexample will show that a statement is true or false, respectively.
    a. existence proof
    b. proof by example or proof by construction
    c. proof by exhaustive example (a long winded example or "proof by cases")
    d. proof by counter example
2. Proof by Contradiction -- We use this type of proof to show that a statement is TRUE, indirectly using the axioms of logic. We use this type of proof when an exhaustive example is not feasible -- for example there may be an infinite number of cases.
3. Proof by Contrapositive -- Here we show that a contrapositive of a statement is true thus showing that the original statement is true. It is used in similar proof situations as proof by contradiction.
4. Resolution Proof -- This is a proof by direct logical deduction and is arguably a type of direct proof. This proof type might be used when a series of logical statements is presented and a conclusion is desired.
5. Proof by induction -- Induction is an advanced proof technique that relies on the following logic (which is a two or sometimes considered 3 step process -- a basis step and induction step): we show that a statement is true by first showing it is true for any random integer directly (we typically choose a small number which helps us show the statement is true for a larger set of numbers). Then we proof that IF our statement is true for an arbitrary integer, then it is true for the number (integer) after the arbitrary one. This is also stated, "If it is true for n, then it is true for n+1". **Note** *this final step must be general and not specific to the arbitrary number and therefore best to use a variable such as n.*

    Intuitive interpretation: Well since we have shown the statement is true for at least one integer and we have shown that --if the statement is true for one number then it is true for the number after it, then we can propogate this logic and show it is true for the integer after that, and the integer after that, ... and ultimately conclude it is true for all integers after the initial number we directly proved.

    This type of proof is used when there is clear mapping to the integers or a countable subset thereof within the statement we are attempting to prove.

TIP: When a proof methods seems unpromising ... try another. I suggest starting with direct proof and working your way down since direct proofs are typically easy.

TIP: There are a number of GREAT examples in this chapter! Use them.

Example:

Goal: Show that if m is even then m^2 is even.

Analysis: Since we can represent even numbers simply, we might be able to prove this directly using simple arithmetic. So lets try and proceed by direct proof.

Steps: Directly show that the statement is true.

If m is even then there exists a number n such that m = 2n**.** Therefore m^2 = 4n^2 = 2(2n^2). And thusly, m^2 is even.

QED.

Example:

Goal: Show that there exists an x such that x+x = x^2

Analysis: Since we only need to show existence of a solution, then it suits us to try an existence proof!

Steps: Show that a solution exists.

Try x = 2. 2+2 = 2*2. Therefore a solution exists. QED.

Example:

Goal: Show that the following statement is false: x+x = x^2 . (Note that logically speaking a statement is true only if it is ALWAYS true. In our previous example we used a existential quantifier).

Analysis: Well, its best to try some numbers here to see if this statement is true often or NOT. We know there exists a solution from our previous example, but ... is this statement always true. I think Not! *Therefore, the only thing we need to do to show that a statement is NOT TRUE is to show a counter example!*

Steps: Show the counterexample

Try x = 3. 3+3 != 3*3. Therefore the statement is not true. QED.

Example: 2.2.1

Goal: Show that for any real number x, if x^2 is irrational then x is irrational

Analysis: Direct proof seems impractical since we cannot show this is true for ALL numbers exhaustively -- there are an infinite number of them. We cannot use a proof by induction since we must show this statement is true for all real numbers and not just the integers. *When this is the case -- we typically rely on proof by contradiction!*

Steps: Assume the antecedent is true and the COMPLEMENT of the consequent is true. Then show that we reach a contradiction, which is logically the same as showing that the original proposition is true (Don't believe it ... try a truth table).

Assume x^2 is irrational and x is RATIONAL. If x is rational then there exists two integers p and q such that x = p/q. This means that x^2 = (p^2)/(q^2). Which means that x^2 is rational, BUT we assumed that x^2 is irrational. We have reached a contradiction and thus our proof by contradiction is done. We have shown that that for any real number x, if x^2 is irrational then x is irrational.

QED

Example:

Goal: Show that 1/(2^2-1)+1/(3^2-1)+...+1/((n+1)^2-1) = 3/4 - 1/(2(n+1))-1/(2(n+2)) is true for all integers greater than or equal to 4.

Analysis: We cannot use a direct proof here since we cannot further manipulate this equation to simply it and we cannot use a proof by cases since we would need to show that this statement is true for ALL Integers which is impractical. Therefore we will need to proceed with either proof type 2,3 or 5. Then we notice that our statement is asking us to <u>proof the above statement for integers</u> greater than 4 -- *this seems like a perfect fit for proof by induction!*

Basis step: n = 4. Show it is true for some small number.

1(2^2-1)+1(3^2-1)+...+1((n+1)^2-1) = .5667

3/4 - 1/(2(n+1))-1/(2(n+2)) = .5567

Induction Step: (Side note: The induction step is sometimes considered two steps: assumption step and induction step)

Assume this equation is true for n, then show that it holds for n+1. that is ...

Assume 1/(2^2-1)+1/(3^2-1)+...+1/((n+1)^2-1) = 3/4 - 1/(2(n+1))-1/(2(n+2)) , show that 1/(2^2-1)+1/(3^2-1)+...+1/(((n+1)+1)^2-1) = 3/4 - 1/(2((n+1)+1))-1/(2((n+1)+2)). Remember, simply plug in n=1 into n. Now we show that the left hand side LHS is equal to the RHS of the equation.

1/(2^2-1)+1/(3^2-1)+...+1/(((n+1)+1)^2-1) =

3/4 - 1/(2(n+1))-1/(2(n+2)) +1/((n+1)+1)^2-1) = (Using our assumption)

3/4 - 1/(2(n+1))-1/(2(n+2)) +1/((n+2)^2-1) =

3/4 - 1/(2(n+2))- ((n+2)^2-1) /(2(n+2) ((n+2)^2-1)) +2(n+2) /(((n+2)^2-1) 2(n+2)) = (Multiply numerator and demon to get similar denoms in last two terms)

3/4 - 1/(2(n+2))   [- ((n+2)^2-1) +   2(n+1)] /(2(n+1) ((n+2)^2-1)) = (Combine numerators)

3/4 - 1/(2(n+2))   [-n^2 +   -2n -1] /(2(n+1) ((n+2)^2-1)) = (Simplify the numerator)

3/4 - 1/(2(n+2))   [-(n+1)^2] /(2(n+1) ((n+2)^2-1)) = (Factor numerator)

3/4 - 1/(2(n+2))   [-(n+1)^2] /(2(n+1) (n^2 + 4n + 3)) = (expand denom)

3/4 - 1/(2(n+2))   [-(n+1)^2] /(2(n+1) (n+1)(n+3)) = (factor denom in hopes of finding an n+1 term for canceling ... we do!)

3/4 - 1/(2(n+2))   [-1] /(2 ((n+3))) = (cancel similar terms in numerator and denom)

Therefore we have shown that the equation holds for n+1 (assuming it is true for n). Therefore our proof is complete.

QED

Another Example 2.4.7:

Problem 2.4 #7:

Using Induction, verify that each equation is true for every positive integer n

$$\frac{1}{1.3}+\frac{1}{3.5}+....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Solution:

According to the principle of Mathematical Induction, if the equation is true for n=1, then for all n>=1, if the equation (S(n)) is true , then the equation (S(n+1)) is true.

Let's prove that the equation will be true for n=1:

$$\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{2(1)+1}$$

$$\frac{1}{1*3}=\frac{1}{3}$$
$$\frac{1}{3}=\frac{1}{3}$$

So this is true for n =1.

Now let's assume the equation is true for n;

Therefore $S_n = \frac{1}{1.3}+\frac{1}{3.5}+....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

$$\frac{1}{1.3}+\frac{1}{3.5}+....+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2n+1)(2n+3)}=$$
$$\frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}=\frac{n(2n+3)}{(2n+1)(2n+3)}+\frac{1}{(2n+1)(2n+3)}=\frac{n(2n+3)+1}{(2n+1)(2n+3)}=\frac{2n\text{^}2+3n+1}{(2n+1)(2n+3)}=\frac{(2n+1)(n+1)}{(2n+1)(2n+3)}=\frac{(n+1)}{(2(n+1)+1)}$$

Therefore this statement holds for n+1 as well!

QED